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Dynamical Algebras in Classical Mechanics

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For a spectrum-generating algebra of classical observables, it is proven that the phase space dynamics simplifies to a Hamiltonian system on submanifolds of the algebra's dual. These submanifolds are coadjoint orbits if the algebra arises from a symplectic group action. If the Hamiltonian splits into the sum of a function of the algebra generators plus a commuting part, then the dynamics transfers to the dual space and an explicit formula is given for the flow vector field on the coadjoint orbits. A unique feature of the presentation is that all constructions are at the Lie algebra level.

1. INTRODUCTION

Symmetry plays an important role in simplifying the equations of motion that arise in physics. In classical mechanics, symmetry gives certain observables, such as energy and angular momentum, which are constant on the phase space orbits of the system. The behavior of these observables often provides the crucial information needed to understand the system. In quantum mechanics, symmetry helps in a different way. Symmetry tells us that the problem of solving Schrödinger's equation can be reduced to considering states that occur in irreducible representations of the symmetry group. Moreover, this reduction enables one to extract information about the expectation values and matrix elements of the symmetry operators in the quantum system.

It was realized early in the development of quantum mechanics that part of the idea involved in the application of symmetry could still be used even when there was no symmetry in the system. This idea is that under certain circumstances one can determine the time evolution of some of the important observables in the problem without actually finding a complete

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description of the physical state in question. This idea is particularly crucial in nuclear physics, where the complete n-body system has a time evolution that is exceedingly complex. In one of the first models of nuclear collective motion, Elliott (1958) suggested investigating a class of observables which provide a measure of the general shape of the nucleus. He used the total orbital angular momentum and the quadrupole tensor as the relevant collective observables. These observables close under commutation to form an 8-dimensional Lie algebra isomorphic to $su(3)$. Although it is not unreasonable to assume that the angular momentum of the system arises from symmetry, it is completely unnatural to assume that the quadrupole tensor does. The attention on the quadrupole tensor arises from its importance as a collective observable and not from any symmetry considerations. However, one can make a natural physical assumption about the problem that does enable one to determine the dynamics of these observables. Rather than assuming the Hamiltonian enjoys some symmetry property with respect to the special observables, it is enough to assume that the Hamiltonian is simply a function of these observables alone. Under this assumption the problem can again be reduced to an irreducible representation of $su(3)$, even though there is no symmetry involved. A number of different applications of these ideas have been made in atomic (Barut and Kleinart, 1967), condensed matter (Sturge, 1967), nuclear (Goshen and Lipkin, 1959; Rowe, 1985; Buck *et al.*, 1979; Iachello and Arima, 1987), and particle physics (Guralnik *et al.*, 1968; Beg and Sirlin, 1974). A Lie algebra of observables that bears this relation to the Hamiltonian is called a dynamical algebra (Barut *et al.*, 1986).

The question that we would like to address in this paper is how to give a precise formulation to a similar idea in classical mechanics. We assume that we are given a collection of physical observables which forms a Lie algebra with the Poisson bracket. Our objective is to find the dynamics of these observables. It turns out that as long as the Hamiltonian is a function of these observables alone, we can simplify the dynamics considerably. There are symplectic manifolds associated with the Lie algebra of observables, called coadjoint orbits, the points of which correspond to all the possible values the observables in the Lie algebra can take on. Moreover, there is a Hamiltonian on each coadjoint orbit which corresponds to the original Hamiltonian and which determines exactly the time evolution of all of the observables in the Lie algebra. Thus the coadjoint orbits of the Lie algebra play a role analogous to the role played by irreducible representations in quantum mechanics. Since the coadjoint orbits always have dimension less than the dimension of the Lie algebra of observables itself, this reduction usually results in a much simpler problem.

In Section 2 we discuss the dynamics of the observables in general. If we do not insist that the dynamics of the observables be Hamiltonian, then a slightly weaker condition on the Hamiltonian is needed. This is simply that the Poisson bracket of the Hamiltonian with every observable in the algebra be a function only of observables in the algebra. This condition has arisen in quantum mechanics, and a Lie algebra of observables that satisfies this condition is called a spectrum-generating Lie algebra (Dothan *et al.*, 1965; Dothan, 1970; Joseph, 1974).

In Section 3 we give the basic theorem, which states when the dynamics of the total phase space translates to Hamiltonian dynamics on the coadjoint orbits. The basic result is Theorem 3.6, which states that if the Hamiltonian is a sum of one part which is a function of the special observables only together with another part which commutes with all the special observables, then the dynamics transfers to the coadjoint orbits.

In Section 4 we apply these ideas to some examples.

2. DYNAMICS ON LIE ALGEBRAS OF OBSERVABLES

We start by establishing the notation for the basic concepts that we will use in the paper.

Notation 2.1:

(a) P is a symplectic manifold with symplectic form ω.

(b) F(P) is the set of observables on P; that is, the set of smooth, real-valued functions on P.

(c) If θ is a one-form on P, then $\theta^{\#}$ is the vector field defined by the relation

$$\omega(\theta^{\#}, v) = \theta(v) \qquad \forall v \in T(P)$$

(d) Let $f, g \in \mathrm{F}(P)$. The Poisson bracket $\{f, g\}$ is defined by

$$\{f, g\} = \omega(df^{\#}, dg^{\#}) = dg^{\#}(f) = -df^{\#}(g)$$

F(P) is a Lie algebra with multiplication defined by the Poisson bracket.

(e) a is a Lie subalgebra of F(P). $\mathrm{a}^* = \mathrm{Hom}(\mathrm{a}, \mathrm{R})$ is the dual vector space to a.

(f) Let $\{f_i\}$ be a vector space basis for a. Define coordinates x_i for a^* by $x_i(g^*) = g^*(f_i)$; that is, $x_i = f_i^{**}$.

(g) Define c_{ij}^k by $\{f_i, f_j\} = \sum_k c_{ij}^k f_k$.

(h) F(a) denotes the set of real-valued functions on a.

Definition 2.2:

(a) $\mathrm{p}: P \to \mathrm{a}^*$ is defined as follows:

$$\mathrm{p}(p)(f) = f(p)$$

p is called the moment map. Using the coordinates of Notation 2.1(f), we have $\mathrm{p}(p)=(f_1(p), f_2(p), \ldots, f_n(p))$, or in other words, $x_i \circ p = f_i$.

(b) Define p^*: $\mathrm{F}(\mathrm{a}^*) \to \mathrm{F}(P)$ by

$$\mathrm{p}^*(g)(p) = g(\mathrm{p}(p))$$

If we write g in coordinates as $g(x_1, x_2, \ldots, x_n)$, we have

$$\mathrm{p}^*(g)(p) = g(f_1(p), f_2(p), \ldots, f_n(p)).$$

Next we give a natural assumption for the dynamics on P to transfer over to the dynamics on a^* by way of the moment map.

Definition 2.3. Let h be the Hamiltonian on P. a is called h-spectrum generating if $\{h,f\} \in \mathrm{p}^*(\mathrm{F}(\mathrm{a}^*))$ for every $f \in \mathrm{a}$.

Now, if a is an h-spectrum-generating algebra, we have that there are functions $g_i \in \mathrm{F}(\mathrm{a}^*)$ such that

$$\dot{f}_i = -\{h, f_i\} = g_i(f_1, \ldots, f_n)$$

This suggests that the vector field H on a^* defined by

$$H = \sum g_i \, \mathrm{d}/\mathrm{d}x_i$$

should correspond to the Hamiltonian vector field $dh^{\#}$.

Theorem 2.4. Let a be an h-spectrum-generating algebra. Then, on $\mathrm{p}(P)$ we have that

$$D\mathrm{p}(dh^{\#}) = H$$

where $H = \sum g_i \, \mathrm{d}/\mathrm{d}x_i$ and $-\{h, f_i\} = \mathrm{p}^*(g_i)$.

Proof. We show this equality by showing that both $D\mathrm{p}(dh^{\#})$ and H give the same result when they differentiate the coordinate functions x_i. We have

$$D\mathrm{p}(dh^{\#})(x_i)(\mathrm{p}(p)) = dh^{\#}(x_i \circ \mathrm{p})(p) = dh^{\#}(f_i)(p) = -\{h, f_i\}(p)$$

We also have

$$H(x_i)(\mathrm{p}(p)) = g_i(\mathrm{p}(p)) = (\mathrm{p}^* g_i)(p) = \{h, f_i\}(p)$$

which completes the proof. ■

The following is an important special case of h-spectrum-generating algebras.

Theorem 2.5. If $h \in p^*(F(a^*))$, then a is an h-spectrum-generating algebra. In particular, if $h' \in F(a^*)$ is any function with $h = h' \circ p$, then

$$Dp(dh^{\#}) = H$$

on $p(P)$, where

$$H = -\sum [dh'/dx_j] c_{ji}^k x_k \, d/dx_i$$

Proof. Let $h = p^*(h') = h' \circ p$. First we calculate $\{h, f_i\}$ using the fact that a is an algebra and hence $\{f_j, f_i\} = \sum c_{ji}^k f_k$ for some constants c_{ji}^k:

$$\begin{aligned} \{h, f_i\} &= dh(df_i^{\#}) \\ &= dh' \circ Dp(df_i^{\#}) \\ &= \sum [dh'/dx_j]\, dx_j \circ Dp(df_i^{\#}) \\ &= \sum [dh'/dx_j]\, d(x_j \circ p)(df_i^{\#}) \\ &= \sum [dh'/dx_j]\, df_j\,(df_i^{\#}) \\ &= \sum [dh'/dx_j]\{f_j, f_i\} \\ &= \sum [dh'/dx_j]\, c_{ji}^k f_k \\ &= p^*(\textstyle\sum [dh'/dx_j] c_{ji}^k x_k) \qquad (*) \end{aligned}$$

Thus $\{h, f_i\} = p^*(g_i)$ for the $g_i = \sum [dh'/dx_j] c_{ji}^k x_k$. We now find that a is an h-spectrum-generating algebra by observing that $\{h, \sum a^i f_i\} = p^*(\sum a^i g_i)$. We obtain the formula for H by using (*) together with Theorem 2.4. This completes the result. ■

3. HAMILTONIAN DYNAMICS ON A LIE ALGEBRA OF OBSERVABLES

In this section we will show that if there is an h' such that $h = h' \circ p$ [i.e., if $h \in p^*(F(a^*))$], then the H defined in Theorem 2.5 will be a Hamiltonian vector field on certain submanifolds of a^*. We call these submanifolds coadjoint orbits because they would correspond to the coadjoint orbits of a Lie group if a arises as the algebra of observables associated with a symplectic action of the group on P (Kirillov, 1962; Kazhdan *et al.*, 1978; Sternberg, 1975). Although there always is a Lie group G which corresponds to a, there may not be any action of G on P because the vector fields $df^{\#}$ for $f \in a$ may not be complete.

There does not seem to be a clear physical reason why such a group action should be necessary to discuss the dynamics of the observables in a.

Fortunately the mathematics needed to relate homogeneous symplectic G manifolds to coadjoint orbits works perfectly well on the algebra level without recourse to groups. In fact, the mathematics even becomes somewhat easier, and we include what we need below for the sake of completeness. The only notable feature which is different in the algebra case is that the image of P under the momentum map may only contain part of a given coadjoint orbit and so the Hamiltonian on a coadjoint orbit which corresponds to h need not be uniquely defined on the whole orbit. This point has no effect on the results concerning the dynamics; it serves only to warn us to include assumptions about being inside p(P) in our results.

We start by defining coadjoint orbits without recourse to any group.

Definition 3.1. For each $f=\sum_i a^i f_i \in \mathrm{a}$ define the following vector field on a^*:

$$\mathscr{L}_f = \sum_{i,j,k} a^i c_{ij}^k x_k \, \mathrm{d}/\mathrm{d}x_j$$

(Recall $\{f_i, f_j\} = \sum_k c_{ij}^k f_k$.)

Lemma 3.2. We have

$$[\mathscr{L}_f, \mathscr{L}_g] = \mathscr{L}_{\{f,g\}}$$

Proof. The proof of this is a straightforward computation using the Jacobi identity for $\{\cdot, \cdot\}$. ■

This lemma implies that the distribution $D = \{\mathscr{L}_f : f \in \mathrm{a}\}$ is involutive. The Frobenius theorem then states that a^* is the disjoint union of submanifolds each of which has D as its tangent space. These submanifolds are the coadjoint orbits of the corresponding group action. We establish a notation for them below.

Definition 3.3:

(a) If $F \in \mathrm{a}^*$, we let O_F denote the maximal integral submanifold of D. O_F is called a coadjoint orbit. If we do not need to specify that a coadjoint orbit contains some specific F, we may denote the orbit simply by O.

(b) Let O be a coadjoint orbit. Define a two-form ω^* on O as follows:

$$\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F) = F(\{f, g\}) \qquad \forall F \in \mathrm{O}$$

The following lemma gives us the basic tools that we need.

Lemma 3.4:

(a) Let $f = \sum a^i f_i$, $g = \sum b^j f_j$, and $F = (x_1, \ldots, x_n)$. Then

$$\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F) = \sum c_{ij}^k x_k a^i b^j$$

(b) ω^* is a symplectic form on O.

(c) Again let $f = \sum a^i f_i$. Using the symplectic form ω^*, we find that

$$\left(\sum a^i\, dx_i\right)^{\#} = -\mathscr{L}_f$$

and hence,

$$dx_i^{\#} = -\sum_{j,k} c_{ij}^k x_k\, \mathrm{d}/\mathrm{d}x_j$$

Proof. Part (a) follows directly from the definition of ω^*. To see (b), we must show that ω^* is well defined, nondegenerate, and closed. To show that ω^* is well defined it is enough to show that $\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F)$ is 0 if $\mathscr{L}_f(F)$ is 0. When $\mathscr{L}_f(F) = 0$ we have

$$\sum_{i,k} a^i c_{ij}^k x_k = 0 \qquad \forall_j$$

so

$$\sum_{i,j,k} a^i c_{ij}^k x_k b^j = 0$$

as desired. Next we show nondegeneracy. Assume

$$\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F) = 0 \qquad \forall g \in \mathrm{a}$$

Then we have

$$\sum c_{ij}^k x_k a^i b^j = 0 \qquad \forall b^j \in \mathrm{R}$$

Thus $\sum c_{ij}^k x_k a^i = 0$, $\forall j$, so that

$$\mathscr{L}_f = \sum_{i,j,k} a^i c_{ij}^k x_k\, \mathrm{d}/\mathrm{d}x_j = 0$$

as desired. We must only show ω^* is closed. Let $h = \sum_i c^i f_i$. A direct computation gives the following two formulas. Use the formulas to simplify the six terms in $d\omega^*(\mathscr{L}_f, \mathscr{L}_g, \mathscr{L}_h)$:

$$\mathscr{L}_f \omega^*(\mathscr{L}_g, \mathscr{L}_h)(F) = \sum_{\substack{i,j,k \\ l,m}} c_{im}^l c_{jk}^m a^i b^j c^k x_l$$

and

$$\omega^*([\mathscr{L}_f, \mathscr{L}_g], \mathscr{L}_h)(F) = \sum_{\substack{i,j,k \\ l,m}} c_{mk}^l c_{ij}^m a^i b^j c^k x_l$$

Then use the Jacobi identity two times to find $d\omega^*=0$. We finish with (c). We have from (a)

$$\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F)=\sum c_{ij}^k x_k a^i b^j$$

Now

$$-\sum a^i\, dx_i(\mathscr{L}_g)(F)=-\sum a^i\, dx_i\left[\sum_{i,j,k} b^i c_{ij}^k x_k\, \mathrm{d}/\mathrm{d}x_j\right]$$
$$=-\sum c_{ij}^k x_k b^i a^j$$
$$=\omega^*(\mathscr{L}_f, \mathscr{L}_g)(F)$$

The proof is now complete. ■

We will now show that if $h \in \mathrm{p}^*(\mathrm{F}(\mathrm{a}^*))$, then not only will H be tangent to every coadjoint orbit O, but H will also be a Hamiltonian vector field on O.

Theorem 3.5. Let $h=h' \circ \mathrm{p}$, O be a coadjoint orbit, and use ω^* to define $\#$ on O. Then we have

$$H=(dh')^{\#}$$

on $\mathrm{p}(P)$.

Proof. Use Lemma 3.4(c) to find that

$$(dh')^{\#}=\sum\,[\mathrm{d}h'/\mathrm{d}x_j](dx_j)^{\#}$$
$$=-\sum\,[\mathrm{d}h'/\mathrm{d}x_j]\sum_{l,k} c_{jl}^k x_k\, \mathrm{d}/\mathrm{d}x_l$$

Theorem 2.5 now completes the proof. ■

Probably the most interesting theorem from the point of view of applications is not the theorem that assumes that the Hamiltonian is a function only of the special observables, but rather the theorem that applies to Hamiltonians whose only terms that do not depend on the special observables commute with those observables. We state this result precisely below.

Theorem 3.6. Let $h=h_0+h_1$, where $h_0=h_0' \circ \mathrm{p}$ and $\{h_1, f\}=0$ for all $f \in \mathrm{a}$.

Then

$$H=(dh_0')^{\#}=-\sum_{i,j,k}\,[\mathrm{d}h_0'/\mathrm{d}x_i]c_{ij}^k x_k\, \mathrm{d}/\mathrm{d}x_j$$

Proof. Since $\{h_1, f\}=0$ for all $f \in \mathrm{a}$, we have that a is both an h-algebra and an h_0-algebra. Moreover, H (corresponding to h) will be the same as H_0 (corresponding to h_0) (see Theorem 2.4). Thus the result follows from Theorem 2.5. ■

4. APPLICATIONS

We would like to start this section with a discussion of the relativistic free particle, which is one of the problems in which coadjoint orbits have already been seen to play a fundamental role (Souriau, 1970). If the classical configuration space for the relativistic free particle is taken to be Minkowski space, then all the quantum states obtained by the conventional quantization procedure have spin 0. Thus, it seemed that the quantum description of a free particle was not the "quantization" of any classical system. However, it was discovered that if one quantizes the classical systems defined on the coadjoint orbits of the Poincaré group, then one obtains the full quantum description of the free particle with arbitrary spin. The condition that the classical systems on the orbits can be quantized corresponds exactly to the Bohr–Sommerfeld quantization condition. Thus the quantum relativistic free particle does arise as the quantization of classical systems. The only question that remains is why should the coadjoint orbits of the Poincaré group be the symplectic manifolds relevant to the free particle? One answer to this question is that it is natural for the phase space of a free particle to be a symplectic manifold which is homogeneous under the action of the Poincaré group, and every such manifold is a covering space of a coadjoint orbit. This is the classical form of the criterion applied by Wigner (1939) to determine the quantum Hilbert spaces for free particles from the irreducible unitary representations of the Poincaré group.

The ideas in this paper give rise to a different way to answer this question. In trying to describe the free particle, we do not attempt to actually describe its configuration space, because it may have internal degrees of freedom in which we are not interested. Instead we focus on the aspects of the free particle we do wish to look at. In this case it seems natural to try to describe the particle's momentum and angular momentum. We do not need to know the exact form these observables take in the problem; we only need to know what their commutation relations are under the Poisson bracket. The natural physical assumption is that these observables form the Lie algebra of the Poincaré group. As of yet, we have not used anything that is specific to the free particle. The fact that we are dealing with a free particle gives information about the Hamiltonian. Normally the Hamiltonian of a free particle is a function of the momenta alone. Thus it is not unnatural to assume that the free particle Hamiltonian is a function of the observables in the Poincaré Lie algebra, and we may use Theorem 3.5 to reduce the dynamics to the coadjoint orbits of the Poincaré Lie algebra. In this way, one arrives at the classical symplectic manifolds which are needed to give a full quantum description of the relativistic free particle.

A second interesting example is the classical theory of rotating self-gravitating fluids formulated by Dirichlet and Riemann and applied to rotating stars by Chandrasekhar (1969) and Dyson (1968). A Riemann ellipsoid is a fluid with an ellipsoidal boundary whose motion depends linearly on position. The observables characterizing a Riemann ellipsoid span the Lie algebra $gcm(3)$, which signifies the general collective motion algebra in three spatial dimensions (Buck *et al.*, 1979; Rosensteel and Ihrig, 1979; Guillemin and Sternberg, 1980). The algebra $gcm(3) \equiv [\mathrm{R}^6]gl(3, \mathrm{R})$ is a semidirect sum of an Abelian six-dimensional ideal spanned by the symmetric inertia tensor Q_{ij} with the Lie algebra of the general linear group $gl(3, \mathrm{R})$, whose generators form the shear tensor N_{ij}, $1 \leq i, j \leq 3$.

The linear velocity field is determined uniquely by the value of the shear tensor. The lengths and directions of the ellipsoid's principal axes are specified by the eigenvalues and eigenvectors of the inertia tensor.

To reduce the dynamics of Riemann ellipsoids to $gcm(3)$ coadjoint orbits, the Hamiltonian must be a function of the model observables. The potential self-energy V of gravitational attraction is a function of the lengths of the principal axes and, hence, is a rotational scalar in the inertia tensor. Furthermore, the kinetic energy T for a linear velocity field was shown by Cusson (1968) to be a function of the $gcm(3)$ observables, $T = \frac{1}{2}\,\mathrm{tr}({}^t\mathrm{N} \cdot \mathrm{Q}^{-1} \cdot \mathrm{N})$. Thus, the dynamics is Hamiltonian on the coadjoint orbit space. Note that the $gcm(3)$ problem is a special case of Hamiltonian dynamics with semidirect product dynamical groups (Marsden *et al.*, 1984).

Theorem 3.6 is applicable to many problems. The simplest is the motion of the center of mass, for which the relevant algebra is the Heisenberg algebra generated by the position and momentum observables of the center of mass. If the potential energy depends only upon the relative distances among the constituent particles, then the Hamiltonian splits into the sum of the kinetic energy for the center of mass plus the relative Hamiltonian which commutes with the Heisenberg algebra.

A more complicated application of Theorem 3.6 is to n-body problems for which the Hamiltonian splits into a collective part expressible in terms of the generators of a Lie algebra and an intrinsic part which, at least approximately, commutes with the collective algebra observables. The rotational and Bohr–Mottelson nuclear models fall into this class (Villars and Cooper, 1970; Rowe, 1988).

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